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SUBCRITICAL HOPF BIFURCATION AT PHONATION ONSET

J. C. Lucero

Departamento de Matemática, *Universidade de Brasília*, *Brasília DF* 70910 − 900, *Brazil*

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1. INTRODUCTION

From the point of view of the theory of dynamical systems, the onset of the vocal fold oscillation at phonation may be described by a Hopf bifurcation [1, 2]: at a certain bifurcation value of a control parameter, an equilibrium position changes its stability and a limit cycle appears. For example, Titze [3] has shown that a minimum value of the subglottal air pressure (phonation threshold pressure) is required to start the vocal fold oscillation. At this threshold pressure, the initial equilibrium position of the vocal folds becomes unstable, and the vocal fold oscillation appears. Several theoretical studies of the oscillation have used the concept of the Hopf bifurcation, on a variety of mathematical models of the vocal fold system such as the mucosal wave model [3], the one-mass model [4], and two-mass models [5–8].

However, the type of bifurcation (i.e., subcritical or supercritical) has been left unclear. Let one recall briefly that in a supercritical bifurcation, a stable limit cycle is generated as the control parameter passes through its bifurcation value, whereas in a subcritical bifurcation, an unstable limit cycle is absorbed [2]. The supercritical Hopf bifurcation is the simplest way for the onset of an oscillation and it would seem to apply to the vocal folds: e.g., at phonation threshold pressure, the observed vocal fold oscillation (a stable limit cycle) would be generated from the bifurcation. However, experimental evidence suggests that the subcritical case would be more appropriate. In various experimental settings, it has been observed that the biomechanical configuration of the vocal folds at oscillation onset is different from their configuration at oscillation offset. For example, studies of excised larynges [9] and physical models of the vocal fold mucosa [10] have shown that the subglottal pressure is lower at oscillation offset than at oscillation onset. Studies on subjects producing speech have shown that the intraoral pressure is lower at voice onset than at voice offset [11], the airflow is lower [12], the transglottal pressures is higher [13], and the glottal width is smaller [13]. This difference between oscillation onset and offset cannot be explained by a supercritical Hopf bifurcation (which would result in the exact same configuration at oscillation onset and offset). In fact, they suggest the phenomenon of oscillation hysteresis [14]. This phenomenon appears from the combination of a saddle–node bifurcation between limit cycles [2], where a stable and an unstable limit cycle are generated, with a subcritical Hopf bifurcation, where the unstable limit cycle is absorbed. Since oscillation onset and offset occur through different bifurcations

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(the Hopf bifurcation and the saddle–node bifurcation, respectively) and at different values of the control parameter, then the different configurations observed experimentally would result. This phenomenon is also called hard-excitation of an oscillation, in opposition to the soft-excitation, which corresponds to the supercritical bifurcation [1].

In this letter, the existence of a subcritical Hopf bifurcation at oscillation onset will be shown by applying the Hopf Bifurcation Theorem [2] to a simple bidimensional model of the vocal folds, and the subglottal pressure as control parameter considered. Next by deriving a bifurcation diagram by numerical techniques, the phenomenon of oscillation hysteresis will be shown. A more detailed version of this study will be published elsewhere [15].

2. VOCAL FOLD MODEL

A large amplitude version [16] of Titze's mucosal wave model [3], as shown schematically in Figure 1 will be used. One assumes that during the oscillation, the vocal fold tissues propagate a surface mucosal wave in the direction of the airflow. The model is valid for an open glottis only, without collision between the vocal folds during the oscillation cycle. One assumes further that the subglottal pressure P_s is constant during the oscillation cycle, the supraglottal pressure is the atmospheric pressure, and the opposite glottal walls are parallel at the initial (prephonatory) position. The glottal aerodynamics is described following the boundary layer model [17] for high Reynolds numbers, and neglecting pressure losses due to air viscosity.

The model is described by the differential equation

$$m\ddot{x} + r\dot{x} + kx = P_g, \tag{1}$$

where x is the lateral displacement of the vocal fold at the midpoint of the glottis, m, r, and k are the mass, damping, and stiffness, respectively, of the oscillating portion of the vocal fold per unit area of its medial surface, and P_g is the mean glottal air pressure. When the glottal channel along the direction of the airflow

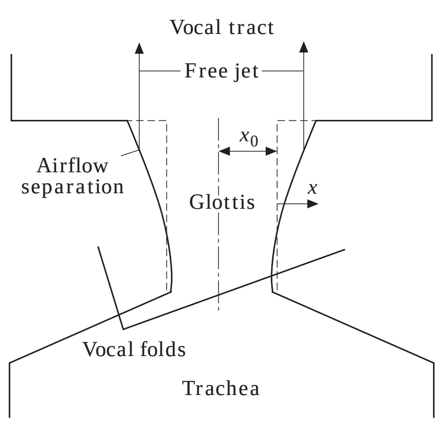

Figure 1. Mucosal wave model of the vocal folds.

is convergent or slightly divergent such that $a_2 \leqslant 1{\cdot}1a_1$, where a_1 and a_2 are the cross-sectional areas at glottal entry and exit respectively, P_g is given by

$$P_g = P_s(1 - a_2/a_1), \qquad (a_2 \leqslant 1{\cdot}1a_1). \tag{2}$$

When the glottal channel is highly divergent, such that $a_2 > 1{\cdot}1a_1$, the airflow detaches from the glottal wall at the point where the glottal area is equal to $1{\cdot}1a_1$ and forms a free jet downstream of the glottis [17]. In this case, P_g becomes

$$P_g = -0{\cdot}01P_s(a_1/(a_2 - a_1)), \qquad (a_2 > 1{\cdot}1a_1). \tag{3}$$

Finally, the glottal areas a_1 and a_2 are given by

$$a_1 = 2L(x_0 + x + \tau\dot{x}), \qquad a_2 = 2L(x_0 + x - \tau\dot{x}), \tag{4, 5}$$

where L is the vocal fold length in the antero–posterior direction, and τ is the delay of the mucosal wave in travelling half the glottal width (in the direction of the airflow). Initially, the vocal folds are at rest at $x = 0$, and $a_1 = a_2$.

Details of the model and the derivation of the above equations may be found in references [3, 15–17].

3. HOPF BIFURCATION AT OSCILLATION ONSET

Letting all the derivatives in equations (1)–(5) equal zero, one finds a unique equilibrium position at the initial position $x = 0$. To investigate its stability, one writes equation (1) as a system of two first order differential equations,

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}), \tag{6}$$

where $\mathbf{u} = (x, \dot{x})$, and analyze the eigenvalues of the associated Jacobian matrix

$$\mathbf{A} = \left(\frac{\partial f_i}{\partial u_j}(\mathbf{0})\right)^2_{i,j=1} = \begin{pmatrix} 0 & 1 \\ -k/m & -r/m + 2\tau P_s/mx_0 \end{pmatrix}. \tag{7}$$

By considering the subglottal pressure P_s as a control parameter, it can be easily shown that two complex eigenvalues cross the imaginary axis transversally from left to right and the equilibrium position becomes unstable at

$$P_x = rx_0/2\tau. \tag{8}$$

At this value of subglottal pressure, the equilibrium position is a weak focus and its Lyapunov number (the first non-zero derivative $\mathrm{d}^{(k)}(0) \neq 0$, where $\mathrm{d}(s) = P(s) - s$ and $P(s)$ is the Poincaré map for the focus) [2] is

$$\sigma = (3\pi r/2\sqrt{mk})(r\tau/m + 3k\tau^2/m + 1) > 0. \tag{9}$$

Since $\sigma \neq 0$, and according to the Hopf Bifurcation Theorem [2], a Hopf bifurcation occurs at the bifurcation value given by equation (8). Further, since $\sigma > 0$, this bifurcation is subcritical.

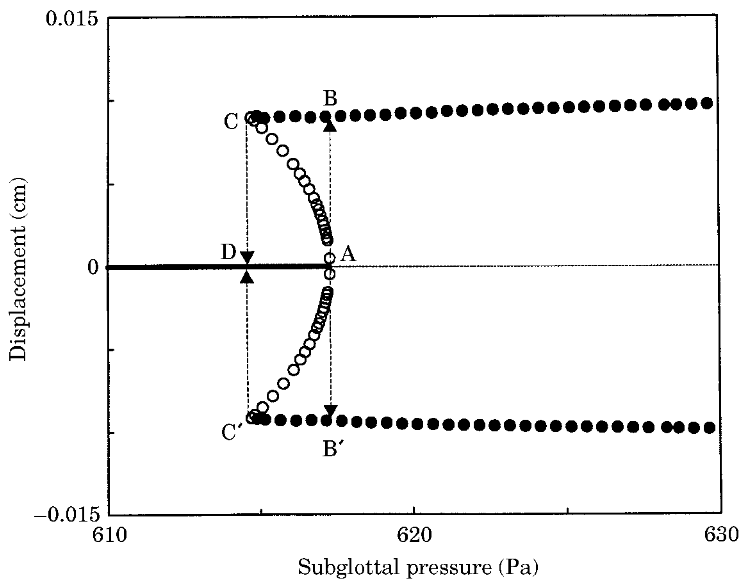

Figure 2. Bifurcation diagram: ——, equilibrium position, stable on the darker portion; ○, maximum and minimum values of an unstable limit cycle; ●, maximum and minimum values of a stable limit cycle. A: subcritical Hopf bifurcation. CC′: saddle-node bifurcation. A–BB′–CC′–D: oscillation hysteresis cycle.

4. OSCILLATION HYSTERESIS

Figure 2 shows a bifurcation diagram for the parameter values $m = 4{\cdot}76$ kg m^{-2}, $r = 1000$ Nsm^{-3}, $k = 2 \times 10^6$ Nm^{-3}, $L = 1{\cdot}4$ cm, $\tau = 0{\cdot}81$ ms, $x_0 = 1$ mm [3]. The diagram was derived using program XPP for solving differential equations in combination with the continuation program AUTO for bifurcation analysis [18]. The horizontal line at $x = 0$ represents the equilibrium position, which is stable on the darker portion of the line. At point A ($P_s = 617{\cdot}2$ Pa) the subcritical Hopf bifurcation occurs: the equilibrium position becomes unstable (to the right of point A) and an unstable limit cycle appears (empty circles). There is also a stable limit cycle (filled circles), which coalesces with the unstable limit cycle at points CC′ in a saddle-node bifurcation (at $P_s = 614{\cdot}7$ Pa).

The vocal folds are initially assumed at rest (at $x = 0$) and the subglottal pressure is increased from 0. The oscillation will start when reaching the Hopf bifurcation at point A, and its amplitude will suddenly increase to points BB′ and will follow next the filled circles to the right. Next, if the subglottal pressure is decreased, the oscillation amplitude will follow the filled circles to the left, until reaching the saddle-node bifurcation at points CC′. The oscillation will then vanish and the vocal folds will return to the rest position at point D. During this process, the system follows the hysteresis path A–BB′–CC′–D. Note that, as a result of the hysteresis, the subglottal pressure is higher at oscillation onset than oscillation offset, in agreement with experimental results [9, 10].

As an additional illustration, Figure 3 shows a phase plane plot for a subglottal pressure $P_s = 616{\cdot}0$ Pa, between both bifurcations. One can see the two limit cycles

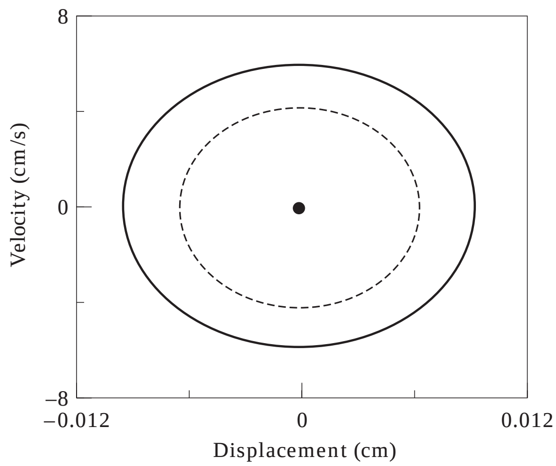

Figure 3. Phase plane plot for $P_s = 616{\cdot}0$ Pa showing two coexistent limit cycles: the internal limit cycle (- - -) is unstable, and the external one (——) is stable. There is also a stable equilibrium position at the origin.

of opposite stability, around the stable equilibrium position at the origin (for clarity, no trajectory was plotted besides the limit cycles).

5. CONCLUSION

This letter has shown that the onset of the vocal fold oscillation may be described mathematically by a Hopf bifurcation of the subcritical type. The combination of this bifurcation with a saddle–node bifurcation between limit cycles results in a oscillation hysteresis phenomenon, which would explain experimental observations of differences in the configuration of the larynx between oscillation onset and offset. This phenomenon appears commonly in cases of flow-induced oscillations (as the vocal fold oscillation); e.g., in the oscillation of buildings and bridges by action of the wind [19].

The analysis was done on a simple bidimensional model of the vocal fold, and considering only the subglottal pressure as control parameter. The simplifications were introduced to reduce the model to its basic principles and provide a qualitative description of its oscillatory dynamics. Further studies of this phenomenon on more elaborated models, such as the popular two-mass model [20], and other control parameters, such as glottal width and vocal fold tension, are desirable as a next step.

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